Damage Localization Using a Statistical Test on Residuals from the SDDLV Approach

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Abstract

Mechanical systems under vibration excitation are prime candidate for being modeled by linear time invariant systems. Damage localization in such systems, when the excitation is not measurable, can be carried out using the Stochastic Dynamic Damage Locating Vector (SDDLV) approach, a method that interrogates changes in a matrix that has the same kernel as the change in the transfer matrix at the sensor locations. Damage location is related to some residual derived from the kernel. Deciding that this residual is zero is up to now done using empirically defined thresholds. In this paper, we describe how the uncertainty of the state space system can be used to derive uncertainty on the damage localization residuals to decide about the damage location. The results are illustrated in finite element models of a truss and of a plate.

Keywords: Ambient vibration; Covariance analysis; Damage localization; Hypothesis testing; Load vectors.

1 Introduction

Alternative approaches to visual inspections of physical structures, such as bridges and buildings, have been provided by vibration-based monitoring techniques. Sensors installed in the structures collect data and statistical approaches originated from stochastic system realization theory for linear systems provide estimates for the parameters of interest. The time evolution of the data is characterized by system matrices of the underlying linear system. The eigenstructure of those matrices relate directly to some parameterization of interest for the monitoring of structures, usually the modal parameters (natural frequencies, damping ratio and mode shapes), and subsequently to the finite element model (FEM). Fault detection (damage detection for mechanical structures) and fault isolation (damage localization) can be inferred from changes in these parameters.

Assuming that damage occurs, \cite{1} presents alternate damage localization techniques using both finite element information and modal parameters: the Stochastic Dynamic Damage Location Vector (SDDLV) approach. From estimates of the system matrices in both reference and damaged states, the null space of the difference between the respective transfer matrices is obtained. Then, damage is related to a residual derived from this null space and located where the residual is close to zero. Empirical thresholds are currently used for decision without considering the intrinsic uncertainty, which happens due to unknown noise excitation and limited data length in the identification of system matrices. The lack of uncertainty consideration is critical: no information is available on the choice of threshold for deciding whether the lowest residual is zero or not in practical situations. Nevertheless, sensitivity based methods, such as presented in \cite{2} and \cite{3}, provide some guidelines to derive uncertainty estimates for modal parameters, and an efficient sensitivity computation of these quantities has been derived in \cite{4, 5, 6}.

This paper aims to replace empirical rules by sensitivity-based rules for applying some damage localization criterion, and is organized as follows. In Section 2 the SDDLV approach is introduced as a method for stochastic damage localization of mechanical structures from output-only signals. In Section 3, the covariance of the system matrices is propagated to the damage localization residuals. In Section 4, numerical examples are provided. Finally, some conclusions of this work are presented in Section 5.
2 The SDDLV approach

The SDDLV, derived in [1, 7], is an output-only damage localization method based on interrogating changes in the transfer matrix $\delta G$ of a system, which is related to a FEM to localize damage without using a detailed model. Typically, the SDDLV is performed in two different situations: one as undamaged (reference) state and another as damaged state. Load vectors of the null space of $\delta G$ are then used for the computation of a stress field over the structure in order to indicate the damage location: Stresses are measures of internal reactions to external forces applied on a deformable body, where (in the method to be described) zero stress over elements of a structure indicates changes in the flexibility and consequently damage. The basic principles and underlying models of the SDDLV are introduced in this section.

2.1 Dynamical equation and state-space model

The behavior of a mechanical structure is assumed to be described by a linear time-invariant (LTI) system and represented by the corresponding continuous-time state-space model

$$\begin{cases} \dot{x} = A_c x + B_c e \\ \eta = C_c x + D_c e \end{cases},$$

where $x \in \mathbb{R}^n$ is the state, $\eta \in \mathbb{R}^r$ is the output, $A_c \in \mathbb{R}^{n \times n}$ is the state transition matrix, $B_c \in \mathbb{R}^{n \times r}$ is the input influence matrix, $C_c \in \mathbb{R}^{r \times n}$ is the output mapping matrix, $D_c \in \mathbb{R}^{r \times r}$ is the direct transmission matrix. The fictive force $e(t)$ acts only in the measured coordinates and that reproduce the measured output, $n$ is the system order and $r$ is the observed outputs coordinates. If all the modes of the LTI system were identified then $n = 2d$. In practice this is seldom the case, so what one gets from identification is a reduced model order $n \ll 2d$. Since SDDLV is an output-only method, the non-identified matrices $B_c$ and $D_c$ are used in order to derive properties of the transfer matrix [1]. Only the system matrices $A_c$ and $C_c$ are relevant for system identification in this paper.

2.2 Damage localization procedure

The damage localization in mechanical structures with output-only data can be determined with the null space vectors for the SDDLV technique. Like this, damage localization information from structural changes (stress over elements) is extracted with the underlying idea of detecting changes in the flexibility. Note that while the transfer matrix is defined at the coordinates defined by the sensors, damage can be localized at any point of the structure because the stress field generated from the sensor coordinate loads covers the full domain.

Consider now the transfer matrix of model (1), which is given by

$$G(s) \overset{\text{def}}{=} R(s)D_c,$$

where

$$R(s) \overset{\text{def}}{=} C_c A_c^{-b}[sI - A_c]^{-1} \left[ \begin{array}{c} C_c A_c^{1-b} \\ C_c A_c^{-b} \end{array} \right] + I$$

with $G(s) \in \mathbb{C}^{r \times r}$, $b = 0, 1, 2$ the output measurements (displacements, velocities, or accelerations respectively) and $I$ the identity matrix.

Using (2) for the damaged (variables with tilde) and reference states, respectively, and dropping the Laplace variables $s$ for simplicity, gives the difference in the transfer matrices $\delta G = \hat{G} - G$. Neglecting $D_c$ in (2) in both damaged and reference states (see [1] for more details), the desired null space of $\delta G$ has the same null space of $\delta R^T = R^T - \hat{R}^T$. Then, the null space of $\delta R^T$ is finally obtained from the Singular Value Decomposition (SVD)

$$\delta R^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^T,$$

where $U, \Sigma, V \in \mathbb{C}^{r \times r}$, $\Sigma_2 \approx 0$ and $V = (v_1, \ldots, v_r) = [V(1) \ V(2)]$ the right singular vectors. Note that $V(1): (v_1, v_2, \ldots, v_t)$ is the nonzero singular vectors and $V(2): (v_{t+1}, v_{t+2}, \ldots, v_r)$ is the ideally zero singular vectors (in
practice small), where a desired load vector \( v \) in the null space of \( \delta R^T \) is then any linear combination of the vectors in \( V(2) \), e.g. \( v = v_r \). For any chosen value \( s \), the load vector \( v = v(s) \) in the null space of \( \delta G(s) \) can be computed as described above, where only model (1) has been used without using information about the geometry of the structure.

The computation of the stress implies knowledge of the geometry of the structure (coming e.g. from a FEM) and is a linear function of displacement \( K \) resulting in matrix \( Q \in \mathbb{R}^{d \times d} \), the transfer matrix \( G_{\text{model}}(s) \equiv (M s^2 + C s + K)^{-1} \) of model of LTI system in the reference state, and the position mapping matrix \( P \in \mathbb{R}^{d \times r} \) with 1’s where each line (position in the structure) and each column (sensor number) agree and zeros elsewhere. Let this function be given by \( \mathcal{L}_{\text{model}}(s) = QG_{\text{model}}(s)P \), such that the stresses \( S(s) \in \mathbb{C}^d \) for a chosen value \( s \) write as

\[
S(s) = \mathcal{L}_{\text{model}}(s)v(s). 
\]

If an element at some degree of freedom \( j \) is damaged, the resulting stress \( S_j(s) \) at coordinate \( j \) from the load \( v(s) \) is zero [1]. Thus, the stresses in \( S(s) \) are considered as damage localization residuals, where the entries close to zero correspond to elements that are potentially (but not necessarily) damaged.

3 Uncertainties on damage localization residuals

System matrices \( A_c \) and \( C_c \) estimated from a finite number of data samples (e.g. using Stochastic Subspace Identification (SSI) methods [8, 9]) are used for the damage localization both in the reference and damaged state. Due to the reduced order model that represents the identified bandwidth, what are obtained is the estimated matrices \( \hat{A}_c \) and \( \hat{C}_c \) and not the “true” system matrices \( A_c \) and \( C_c \). The input of system (1) is unmeasured noise, leading \( \hat{A}_c \) and \( \hat{C}_c \) to variance errors depending on the data and the estimation method. A variance analysis of the system matrices obtained from Stochastic Subspace Identification is made e.g. in [10] and expressions for their computation in the context of structural vibration analysis are given in [3, 5, 6].

When estimating the load vectors in the null space of \( \delta G \) and the related stress field, the uncertainty of the system matrices is propagated to the uncertainty in the damage localization results. In this section, the variances of damage localization results are evaluated in order to support the decision between undamaged and damaged elements. The decision if the stress \( S_j(s) \) at element \( j \) is zero (potentially damaged) or not is facilitated when knowing the variance of the estimate.

Note in this section that first-order matrices and derivation were dropped for simplicity. For further details, see [11, 12].

3.1 Covariance of \( R \)

In this section, the sensitivity of the matrix \( R \) in (3) with respect to the system matrices \( A_c \) and \( C_c \) is derived, which is needed for the damage localization in (4). First, assume that the data is given by acceleration sensors \( b = 2 \). Derivations for displacement and velocity data \( b = 0, 1 \) follow analogously. A perturbation of \( R \) is linked to a perturbation of \( A_c \) and \( C_c \) by the relation

\[
\text{vec}(\Delta R) = [\mathcal{J}_{A_c} \quad \mathcal{J}_{C_c}] \left[ \begin{array}{c} \text{vec}(\Delta A_c) \\ \text{vec}(\Delta C_c) \end{array} \right],
\]

and with notation \( M_{re} \triangleq [\text{Re}(M) \quad \text{Im}(M)]^T \) and (6), the relation

\[
\text{cov}((\text{vec}(R^T))_{re}) = \mathcal{J}_R \text{cov} \left( \begin{array}{c} \text{vec}(A_c) \\ \text{vec}(C_c) \end{array} \right) \mathcal{J}_R^T
\]

holds for the asymptotic covariance of the real and imaginary parts of \( R^T \), where \( \mathcal{J}_R \) is defined as

\[
\mathcal{J}_R = \left[ \begin{array}{cc} P_{rr} & 0_{r^2,r} \\ 0_{r^2,r} & P_{rr} \end{array} \right] \left[ \begin{array}{cc} \text{Re}(\mathcal{J}_{A_c}) & \text{Re}(\mathcal{J}_{C_c}) \\ \text{Im}(\mathcal{J}_{A_c}) & \text{Im}(\mathcal{J}_{C_c}) \end{array} \right].
\]
In order to compute the covariance of the damage localization residual – the stresses $S(s)$ from (5) for a chosen value $s$ –, the covariance of the load vector $v$ is needed, which is a singular vector in the null space of $\delta R^T = R^T - R^T$ in (4). In the following, the first-order perturbation of right singular vectors $v$ in the null space is provided in order to obtain the covariance of the stresses $S(s)_{re}$ in (8).

Let $\text{cov}((\text{vec } R^T)_{re})$ and $\text{cov}((\text{vec } \tilde{R}^T)_{re})$ from the reference and damaged state be given in (7) and the sensitivity in $J_v$. Then,

$$\Sigma_S \overset{\text{def}}{=} \text{cov}(S(s)_{re}) = J_{S(s)} \left( \text{cov}((\text{vec } \tilde{R}^T)_{re}) + \text{cov}((\text{vec } R^T)_{re}) \right) J_{S(s)}^T,$$

where

$$J_{S(s)} = (L_{\text{model}}(s))_{\text{Re}} J_v$$

with $L_{\text{model}}(s)$ defined in Section 2.2.

### 3.3 Hypothesis testing for damage localization

If the stress $\hat{S}_j(s)$ over a finite element $j$ is close to zero, the element is a candidate for being classified as damaged. The values in the stress vector $\hat{S}(s)$ are complex values, whose real and imaginary parts can have different signs. One could for example test if the real parts are close to 0 (neglecting the imaginary part if it is small), or, more general, if both the real and imaginary parts are close to 0 for an element. For each element $j$, this corresponds to the hypotheses

$$\begin{align*}
H_0 & : \hat{S}_j(s) \neq 0 \quad (\text{element is undamaged}) \\
H_1 & : \hat{S}_j(s) = 0 \quad (\text{element is potentially damaged})
\end{align*}$$

(9)

The elements in the vector $\hat{S}(s)_{re}$ are asymptotically Gaussian distributed with non-zero mean under $H_0$ and zero mean under $H_1$. A consistent estimate of the covariance $\Sigma_S$ of $\hat{S}(s)_{re}$ can be obtained from (8). Then, testing $H_0$ against $H_1$ can be done by computing the variables

$$\chi^2_1(1) \overset{\text{def}}{=} \frac{(\text{Re}(\hat{S}_j(s)))^2}{\hat{\sigma}_j^2} \quad \text{or} \quad \chi^2_2(2) \overset{\text{def}}{=} \left[ \begin{array}{c}
\text{Re}(\hat{S}_j(s)) \\
\text{Im}(\hat{S}_j(s))
\end{array} \right]^T \Sigma_\hat{S}^{-1} \left[ \begin{array}{c}
\text{Re}(\hat{S}_j(s)) \\
\text{Im}(\hat{S}_j(s))
\end{array} \right],$$

(10)

where $\hat{\sigma}_j^2 = \Sigma_S(j,j)$ is the entry $(j,j)$ of $\Sigma_S$ and $\Sigma_\hat{S}$ is the covariance of $[\text{Re}(\hat{S}_j(s)) \quad \text{Im}(\hat{S}_j(s))]^T$.

Thresholds $t_1$ and $t_2$ are defined in $\int_0^\beta f_{x^2(i)}(x)dx = 1 - \beta$, where $f_{x^2(i)}(x)$ is the probability density function of the central $\chi^2$ distribution with $i$ degrees of freedom ($i = 1, 2$), and $\beta$ is the probability of deciding that an element is undamaged while it is potentially damaged (type II error of the hypothesis test (9)). Then, using the test $\chi^2_1(1)$, $H_0$ is rejected and $H_1$ is accepted for an element $j$ (damage occurred), if $\chi^2_1(1) \leq t_1$. Using test $\chi^2_2(2)$, $H_0$ is rejected and $H_1$ is accepted for an element $j$, if $\chi^2_2(2) \leq t_2$.

### 4 Numerical application

Two numerical applications using simulated structures were used to validate the damage localization algorithm with hypothesis test from Section 3.3 where both applications have threshold $t_2 = 2.16$ computed at $\beta = 0.34$ to decide if an element is potentially damaged or not (the horizontal line in Figures 2, 3, 5 and 7). Recall that the residual (the stress) is close to zero for damaged elements. Computational time in both applications was a few seconds after the uncertainty computation of the system identification results with elapsed time around 30 seconds in each case.
4.1 Truss structure

The first numerical application uses a simulated 25 DOF truss structure (Figure 1) to validate the damage localization algorithm. Damage was simulated by stiffness reduction on bars. For both the undamaged and the damaged state, a data sample of length \( N = 25,000 \) of acceleration data \( b = 2 \) was generated with added output noise using Gaussian white noise excitation. From the output-only data, the system matrices and their covariances were estimated of the discrete-time state-space system corresponding to (1), using SSI and the uncertainty quantification in [3]. In order to obtain the matrices \( \hat{A}_c \) and \( \hat{C}_c \) of the continuous-time system and their respective covariances, a discrete to continuous transformation was made. The Laplace variable \( s \) was empirically chosen near a pole of \( \hat{A}_c \) to compute the stress \( \hat{S}(s) \) in (5). The covariance \( \hat{\Sigma}_S \) of \( \hat{S}(s) \) was computed from (8), and the corresponding \( \chi^2 \) was computed from (10).

First, the output was generated at six sensor positions in vertical direction at the lower chord (see Figure 1) with 5% output noise added. Damage was simulated by decreasing the stiffness of element 16 by 20%. From system identification, not all of the 25 theoretical modes could be identified at model order \( n = 50 \). Four well-estimated modes were chosen in both the undamaged and the damaged state using a stabilization diagram procedure [13], from where the matrices \( \hat{A}_c \) and \( \hat{C}_c \) and their covariances are obtained in both states. From these system matrices, the real and imaginary parts of the stress values and their covariance are computed for \( s = 2i \). Once the identified modes are selected from the system identification for \( \hat{A}_c \) and \( \hat{C}_c \), the method is automated.

In Figure 2, the values \( \hat{\chi}^2_j(2) \) in (10) are computed on the real and imaginary parts of the stress and their covariance. Two elements are below this threshold: the damaged element 16 as well as the undamaged element 23 that is a neighboring element of 16 (see Figure 1). It can be shown, in fact, that for the used sensor set element 23 is inseparable from element 16 at \( s = 0 \) [14], i.e. if the stress in 16 is zero, so it must be in 23. Although \( s = 2i \) (used here) is not zero, it is small and the noted behavior is clearly manifested. The corresponding \( \chi^2 \)-values are \( \hat{\chi}_{16}^2 = 0.30 \) for element 16 and \( \hat{\chi}_{23}^2 = 1.26 \) for element 23. Thus, the lowest \( \chi^2 \)-value corresponds correctly to the damaged element. The elements 1–15, 17–22 and 24–25 are correctly classified as undamaged. In Figure 2, the \( \chi^2 \)-values are only displayed until the value 300, while some of them were at more than 10^5.

The theory shows that as the number of damaged bars increases the dimension of the theoretical null space decreases and thus the estimation of vectors in the null space under noisy conditions becomes more difficult. In this case, a more precise estimation of \( \delta R_T \) is necessary, which requires more modes to be estimated from the system identification and thus more sensors due to the constraint \( 2r \geq n \). In Figure 3, results are presented for two damages in bars 3 and 18, where twelve sensors (the present six sensors and more six sensors on the upper cord) were used. Twelve estimated mode pairs were chosen with both damaged and undamaged cases. It should be noted that the estimated stresses in the damaged elements are small but different from zero due to modal truncation and noise, which become more important when multiple damages are present. While the resulting \( \chi^2 \)-values of the damaged elements are the lowest and the value of bar 3 is correctly under the threshold, the values of bar 18 exceeds it slightly.

4.2 Plate

The second numerical application was simulated to validate the damage localization algorithm. The plate (Figure 4) nodes and positions “P” were spatially determined to locate undamaged and damaged regions for didactic reasons,
unlike in reality. With 81 DOF, the plate has 150 cm width, 100 cm height and 1 cm of thickness. Edges are fixed and do not have DOF. The output was generated at ten sensor positions in nodes 16, 25, 29, 31, 47, 65, 70, 83, 95 and 98 (green nodes in Figure 4) with added 5% output noise. Downsampling filter on system loads (inputs) limiting the output frequency in 150 Hz was applied to avoid undesirable folding frequencies (Nyquist frequencies) initially generated. From system identification at model order \( n = 30 \), matrices \( \hat{A}_c \) and \( \hat{C}_c \) and their covariances are obtained in both states. From these system matrices, the real and imaginary parts of the stress values and their covariance are computed for \( s = 1 + 140i \). Once the identified modes are selected from the system identification for \( \hat{A}_c \) and \( \hat{C}_c \), the method is automated.

Two simulations were performed on the plate: the first is presented in Figure 5 with damage in element 34 by decreasing the stiffness in 50%. Nine well-estimated modes were chosen in both the undamaged and the damaged state. The values \( \hat{\chi}^2(2) \) in (10) are computed on the real and imaginary parts of the stress and their covariance. Some elements are below the threshold: the damaged element 34 as well as the damage elements around the

Figure 2: \( \chi^2 \)-test values with one damage – 5% output noise, 6 sensors, 20% stiffness reduction in bars 16.

Figure 3: \( \chi^2 \)-test values with multiple damages – 5% output noise, 12 sensors, 20% stiffness reduction in bars 3 and 18.

Figure 4: Plate with ten sensors.
damaged region (see Figure 5). The corresponding \( \chi^2 \)-values is \( \hat{\chi}_34^2 = 0.2 \) for element 34 and is correctly classified as damaged. Neighboring elements with low results such as 25, 35–36, 45–47, 54–58 and 65–66 are near to the damaged region and also classified as damaged. In Figure 5, the \( \chi^2 \)-values are only displayed until the value 6, while some of them were at more than 650.

Alternatively, damage position 34 and damaged region can also be visualized in Figure 6. Each position “P” can be composed by vertical axis as the first number and horizontal axis as the second number (i.e. vertical axis position 4 and horizontal axis position 9 lead to the position P49). The damaged/undamaged scale vary from damaged state value -3 (in red) to undamaged state value 6 (in blue). The damage in position P34 as well as the damaged region are presented in red as expected.

The second simulation on the plate has damage in position P66 by decreasing the stiffness in 50%. Eight well-estimated modes were chosen in both the undamaged and the damaged state. The values \( \hat{\chi}_j^2(2) \) in (10) are computed on the real and imaginary parts of the stress and their covariance. Some elements are below this threshold: the damaged element 66 as well as some elements around (see Figure 7). The corresponding \( \chi^2 \)-values is \( \hat{\chi}_66^2 = 0.60 \) for element 66. The elements 44–46 and 56–57 are located in the damaged region and also classified as damaged. The elements 24–25 are not around the damaged position and in practical cases should be discarded after visual verification. In Figure 7, the \( \chi^2 \)-values are only displayed until the value 6, while some of them were at more than 4000.

The damage position 66 in the plate and the correspondent damaged region is also presented in Figure 8. The damaged/undamaged scale values vary from damaged state value -2 (in red) to undamaged state value 8 (in blue). The damage in position P66 as well as the damaged region are presented in red as expected.

5 Conclusion

Deciding whether a damage localization residual is zero or not is no more based on empirical thresholds, but on uncertainty bounds, which are now obtained for each element that is tested for damage separately, unlike in [1]. Thus, the intrinsic uncertainty from the data is propagated properly for each evaluated element in the damage localization residual \( S(s) \). Then, it can be decided if an element is potentially damaged or undamaged by using hypothesis test that takes into account the uncertainties, which was successfully performed in two numerical applications. Choosing a different identification method could yield a different performance. Future work includes the aggregation of the damage localization residual at different values of the Laplace variable \( s \) using other statistical techniques and the validation of the method on a large-scale example under realistic noisy conditions.
Figure 7: $\chi^2$-test values with one damage – 5% output noise, 10 sensors, 50% stiffness in position P66.

Figure 8: $\chi^2$-test from Figure 7 on a plate representation – damage in P66 (red region).

References


